Chapter 4

Maxwell's Equations: The base of moder Electromagnetism is a set of four equations:

$$\boxed{\begin{aligned} &1)\quad \vec{\nabla}\cdot\vec{D} = \rho_v \\ &2)\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &3)\quad \vec{\nabla}\cdot\vec{B} = 0 \\ &4)\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

E: electric field

$D = \varepsilon E$

B: magnetic field

$B = \mu H$

ρ_v: electric charge density per unit volume

J: Current density

Published in 1873 by James Clerk Maxwell.

In static field all $\frac{\partial}{\partial t}$'s are zero:

Electrostatics: $\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_v \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$

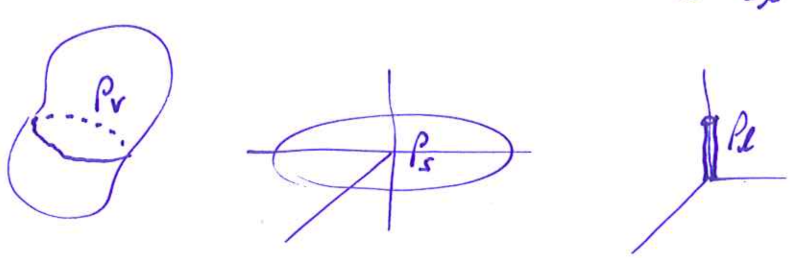
Magnetostatics: $\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{H} &= J \end{aligned}}$

So in static case, E and B are no longer connected.

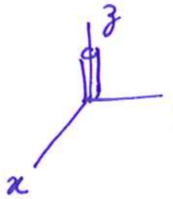
Charge and Current Distributions

charge density:

- Volume charge density $\rho_v = \frac{dq}{dv}$ $\left(\frac{c}{m^3}\right)$ $\rightarrow$ $Q = \int_v \rho_v \, dv$ (c)
- Surface charge density $\rho_s = \frac{dq}{ds}$ $\left(\frac{c}{m^2}\right)$ $\rightarrow$ $Q = \int_s \rho_s \, ds$ (c)
- Line charge density $\rho_l = \frac{dq}{dl}$ $\left(\frac{c}{m}\right)$ $\rightarrow$ $Q = \int_l \rho_l \, dl$ (c)



Ex.: $\rho_\ell = 2z$ on a tube



if $\ell = 10\,cm$, what is the total charge on this tube?

Answer: $Q = \int \rho_\ell \, d\ell = \int_0^{0.10} 2z\, dz = z^2 \Big|_0^{0.1} = 0.01\ C$

Ex. The charge on a disk is linearly increasing with r from zero at center to $6\,C/m^2$ at $r = 3\,cm$. Find the total charge on this disk.

Answer:

$$\rho_s = \frac{6r}{R} \rightarrow Q = \int \rho_s ds = \int \frac{6r}{R} \overbrace{r dr d\varphi}^{ds}$$

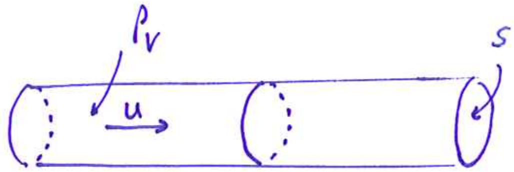
$$Q = \int \frac{6r}{0.03} r dr d\varphi = \int_{\varphi=0}^{2\pi} d\varphi \int_0^{0.03} 200 r^2 dr = 2\pi \times 200 \times \frac{r^3}{3} \Big|_0^{0.03} = 11.31 \times 10^{-3}\ (C)$$

Current density:

Current is the amount of charge moved per unit time: $\boxed{I = \frac{dq}{dt}}$

Current density is J and is current per unit area: $J = \frac{dI}{ds} \rightarrow \boxed{I = \int_s \vec{J} \cdot \vec{ds}}$

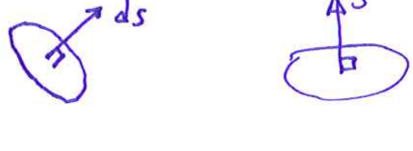
We can also define current for the charges moving in a conductor with velocity u:



$$\left.\begin{aligned} I &= \frac{dq}{dt} = \frac{dq}{dv}\frac{dv}{dt} \\ dv &= s\, d\ell \\ \rho_v &= \frac{dq}{dv} \\ \frac{d\ell}{dt} &= u \end{aligned}\right\} I = \rho_v \frac{s d\ell}{dt} = \rho_v u \rightarrow \underbrace{\frac{I}{s}}_{J} = \rho_v u$$

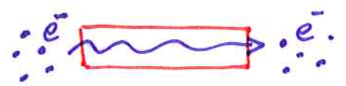
$$\boxed{\vec{J} = \rho_v \vec{u} \quad \left(\frac{A}{m^2}\right)}$$

Note that $\vec{ds}$ and $\vec{s}$ are vectors normal to the surface:



* Convection Current:

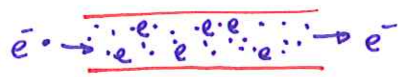
Current is generated by actual movement of charges:



$\rightarrow$ I and V do not obey ohm's law $R \neq \frac{V}{I}$ in general.

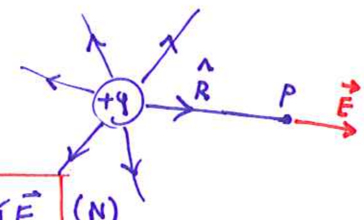
* Conduction current:

Electrons move within atoms and generate current. The electrons that emerge from the wire are not necessarily the same as the one entered the wire at the other end:



$\rightarrow$ I and V obey ohm's law: $R = \frac{V}{I}$

Coulomb's Law



(1) $\vec{E}$ for a charge q is given by: $\boxed{\vec{E} = \hat{R}\,\frac{q}{4\pi\varepsilon R^2}}$

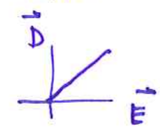
(2) The force $\vec{F}$ on a charge q' in an Electric field $\vec{E}$ is: $\boxed{\vec{F} = q'\vec{E}}$ (N)

$\vec{E}$: Electric field intensity

$\vec{D}$: Electric flux density

D and E are related with $\varepsilon = \varepsilon_r \varepsilon_0$: $\boxed{\vec{D} = \varepsilon \vec{E}}$ where $\varepsilon_0 = 8.85 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}$ $\left(\frac{F}{m}\right)$

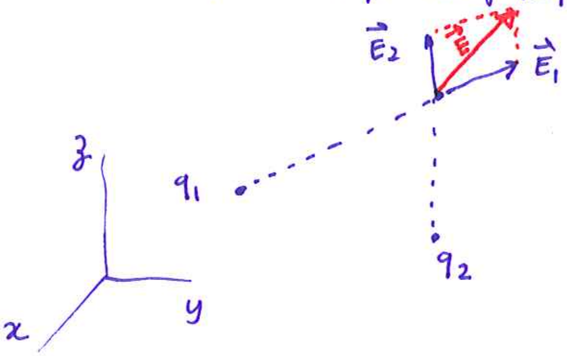
Linear Material: ε is independent of the magnitude of $\vec{E}$. $\vec{D} = \varepsilon \vec{E}$



Isotropic Material: ε is independent of the direction of $\vec{E}$.

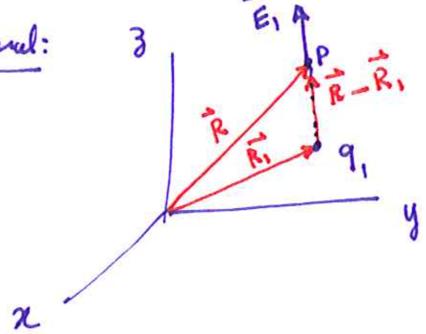
Electric Field due to Multiple Point Charges

The Electric field obeys the principle of linear superposition:



$\vec{E} = \vec{E}_1 + \vec{E}_2$

In general:



$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon\,|\vec{R}-\vec{R}_1|^2}\;\frac{\vec{R}-\vec{R}_1}{|\vec{R}-\vec{R}_1|}$$

$$\boxed{\vec{E}_1 = \frac{q_1(\vec{R}-\vec{R}_1)}{4\pi\varepsilon\,|R-R_1|^3} \quad \left(\frac{V}{m}\right)}$$

R is the vector for point P.

R_1 is the vector for the charge location (q_1 location)

So if we have two charges of q_1 at $\vec{R}_1$ and q_2 at $\vec{R}_2$, the electric field at point $\vec{R}$ is sum of $\vec{E}_1$ and $\vec{E}_2$:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{q_1(R-R_1)}{4\pi\varepsilon\,|\vec{R}-\vec{R}_1|^3} + \frac{q_2(R-R_2)}{4\pi\varepsilon\,|\vec{R}-\vec{R}_2|^3}$$

And if we hav N charges of $q_1, q_2, \ldots q_N$ at locations $R_1, R_2, \ldots R_N$, $\vec{E}$ is:

$$\boxed{E = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(R-R_i)}{|R-R_i|^3}}$$

Example $q_1 = 2\times10^{-5}\,C$ is located at $(1,3,-1)$ and $q_2 = -4\times10^{-5}\,C$ at $(-3,1,-2)$.

Find (a) $\vec{E}$ at $(3,1,-2)$ and (b) the force on a $8\times10^{-5}\,C$ charge located at that point.

Answer:

$$\begin{cases}\vec{R}_1 = \hat{x}+3\hat{y}-\hat{z}\\ \vec{R}_2 = -3\hat{x}+\hat{y}-2\hat{z}\\ \vec{R} = 3\hat{x}+\hat{y}-2\hat{z}\end{cases}$$

(a)

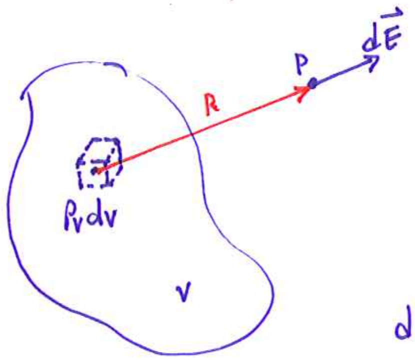
$$\rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(q_1\frac{(\vec{R}-\vec{R}_1)}{|\vec{R}-\vec{R}_1|^3} + q_2\frac{(\vec{R}-\vec{R}_2)}{|\vec{R}-\vec{R}_2|^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{2(2\hat{x}-2\hat{y}-\hat{z})}{(2^2+2^2+1)^{3/2}} + \frac{(-4)(6\hat{x})}{6^3}\right)\times10^{-5}$$

$$= \frac{10^{-5}}{108\pi\varepsilon_0}(\hat{x}-4\hat{y}-2\hat{z})$$

(b) $$\vec{F} = q_3\vec{E} = 8\times10^{-5}\times\frac{10^{-5}}{108\pi\varepsilon_0}(\hat{x}-4\hat{y}-2\hat{z}) = \frac{10^{-10}}{27\pi\varepsilon_0}(2\hat{x}-8\hat{y}-4\hat{z})\ \text{(N)}$$

Electric Field due to a charge Distribution



We want to find the electric fied due to a volume of charge.

If the volume charge density is ρ_v, we can write:

$$d\vec{E} = \hat{R}\,\frac{\overbrace{dq}^{\rho_v dv}}{4\pi\varepsilon R'^2} = \hat{R}\,\frac{\rho_v dv}{4\pi\varepsilon R'^2}$$

$$\rightarrow \quad \vec{E} = \int_v d\vec{E} = \frac{1}{4\pi\varepsilon}\int_v \hat{R}\,\frac{\rho_v dv}{R^2}$$

Note that both R and $\hat{R}$ change with position.

We can write $\vec{E}$ for three cases of volumetric, surface, and line charges:

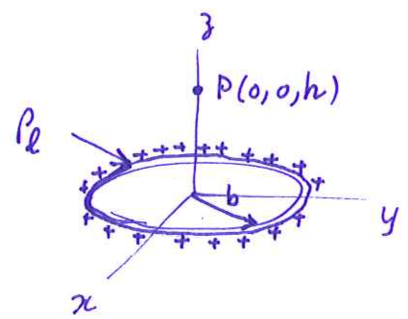
Volume: $\vec{E} = \dfrac{1}{4\pi\varepsilon}\displaystyle\int_v \hat{R}\,\frac{\rho_v dv}{R^2}$

Surface: $\vec{E} = \dfrac{1}{4\pi\varepsilon}\displaystyle\int_s \hat{R}\,\frac{\rho_s ds}{R^2}$

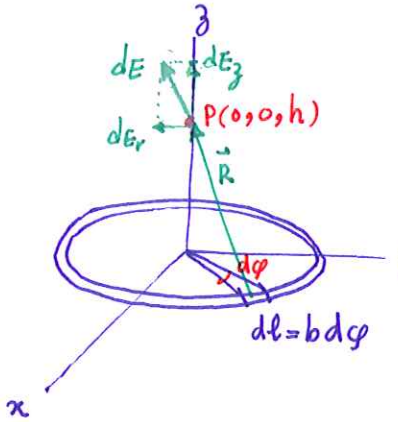
line: $\vec{E} = \dfrac{1}{4\pi\varepsilon}\displaystyle\int_l \hat{R}\,\frac{\rho_l dl}{R^2}$

Example Electric Field of a ring of charge:

Determine the Electric field for a ring of charge with uniform line charge density ρ_l at point $P(0,0,h)$ as shown in the picture:



Solution:



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \hat{R}\,\frac{\rho_\ell\, dl}{R^2}$$

$$\left.\begin{aligned} dl &= b\,d\varphi \\ \hat{R} &= \frac{-b\hat{r}+h\hat{z}}{\sqrt{b^2+h^2}} \end{aligned}\right\} \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\int(-b\hat{r}+h\hat{z})\,\frac{\rho_\ell\, b\,d\varphi}{(b^2+h^2)^{3/2}}$$

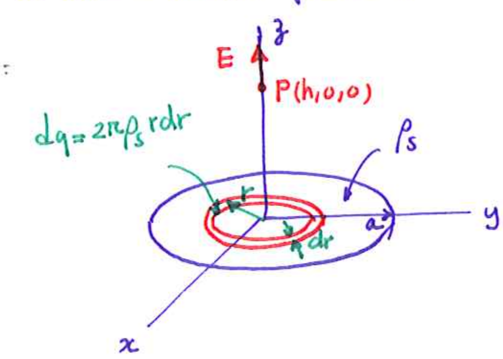
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\int_0^{2\pi}\frac{-b^2\rho_\ell\, d\varphi}{(b^2+h^2)^{3/2}}\,\hat{r}}_{=0 \text{ from symmetry}} + \int_0^{2\pi}\frac{hb\rho_\ell\, d\varphi}{(b^2+h^2)^{3/2}}\,\hat{z}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{hb\rho_\ell}{(b^2+h^2)^{3/2}}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\,\hat{z} = \frac{hb\rho_\ell}{2\varepsilon_0(b^2+h^2)^{3/2}}\,\hat{z}$$

Since the total charge is $Q = 2\pi b\rho_\ell \Rightarrow b\rho_\ell = \frac{Q}{2\pi}$ or $\vec{E} = \frac{hQ}{4\pi\varepsilon_0(b^2+h^2)^{3/2}}\,\hat{z}$

Example Electric Field of a circular disk of charge

Find the electric field at a point $P(0,0,h)$ in free space due to a circular disk of charge in the x-y plane with uniform charge density ρ_s as shown in the picture:



For a ring of radius r and width dr we can write $d\vec{E}$ as:

$$d\vec{E} = \frac{h\,dQ}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}\,\hat{z} \qquad dQ = 2\pi r\,dr\,\rho_s$$

$$d\vec{E} = \frac{\rho_s h}{2\varepsilon_0}\,\frac{r\,dr}{(r^2+h^2)^{3/2}}\,\hat{z}$$

$$\rightarrow \vec{E} = \int d\vec{E} = \hat{z}\,\frac{\rho_s h}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+h^2)^{3/2}} = \hat{z}\,\frac{\rho_s h}{4\varepsilon_0}\int_{h^2}^{a^2+h^2}\frac{du}{u^{3/2}} = \hat{z}\,\frac{\rho_s h}{4\varepsilon_0}\,\frac{-2}{\sqrt{u}}\Bigg|_{h^2}^{a^2+h^2}$$

$$\left(u = r^2+h^2 \rightarrow du = 2r\,dr\right)$$

$$\vec{E} = \hat{z}\,\frac{\rho_s h}{2\varepsilon_0}\left(\frac{1}{\sqrt{h^2}} - \frac{1}{\sqrt{a^2+h^2}}\right) = \hat{z}\,\frac{\rho_s}{2\varepsilon_0}\left(\frac{h}{|h|} - \frac{h}{\sqrt{a^2+h^2}}\right)$$

For infinite sheet of charge $a=\infty \Rightarrow \boxed{\vec{E} = \hat{z}\,\frac{\rho_s}{2\varepsilon_0}}$ (For points with $z<0$, $\hat{z}$ must be replaced by $-\hat{z}$)